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Abstract: We show that an agent maximizing some utility function on a discrete (as opposed to continuous) consumption space will obey the generalized axiom of revealed preference (GARP) so long as the agent obeys cost efficiency. Cost efficiency will hold if there is some good, outside the set of goods being studied by the modeler, that can be consumed by the agent in continuous quantities. An application of Afriat's Theorem then guarantees that there is a strictly increasing utility function on the discrete consumption space that rationalizes price and demand observations in that space.

Keywords: generalized axiom of revealed preference, Afriat's Theorem, discrete demand, utility maximization

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1. Introduction

The revealed preference theory of Afriat (1967), Diewert (1973), and Varian (1982) was developed in the context of a continuous consumption space, typically assumed to be the positive orthant of a Euclidean space. However, the consumption possibilities available to a consumer are often discrete, which gives rise to an "untidy veil" between theory and data.

A basic question that one could ask when given a set of consumer data (of the prices and demand over a set of goods) is whether they are consistent with utility maximization, i.e., whether there is a utility function such that each observation solves the utility maximization problem of the consumer, conditional on the level of expenditure. When the consumption space is continuous, it is well-known that a set of observations is consistent with utility maximization if and only if it obeys the generalized axiom of revealed preference (GARP). However, when the consumption space is discrete, GARP is no longer necessary for consistency with utility maximization, so the continued use of this property in such a context requires a different justification.

In this paper, we show that GARP remains a necessary restriction on the data set, even when the consumption space is discrete, so long as, in addition to utility-maximization, the consumer also takes cost efficiency into account by choosing the cheapest bundle amongst the bundles that give the same utility. We show that in many empirical settings, cost efficiency is a natural assumption to make. This is because an economist studying consumer demand is not typically trying to model the consumer's behavior across the entire range of possible consumption goods. Instead he or she would have data only over some subset $\mathcal{K}$ of goods and would be trying to infer the consumer's preference over goods in $\mathcal{K}$ from demand behavior over those (same) goods. It is well-known that this approach is valid only when the agent's preference on the consumption space of $\mathcal{K}$ is independent of the consumption of goods outside that set; in other words, if the agent's overall utility function, defined on *all* goods, has a separability property. Taking this larger context into account, we show that cost efficiency in the demand for goods in $\mathcal{K}$ is necessary for overall utility maximization, so long as the agent's utility is increasing in some continuous good outside the set $\mathcal{K}$. Therefore, a set of observations of prices and demand for goods in $\mathcal{K}$ from such a consumer *will* obey GARP, and one could then construct a utility function rationalizing those observations using Afriat's Theorem.

2. Violations of GARP in a discrete consumption space

Consider a consumer who chooses from a consumption space X; we assume that X is contained in, but not necessarily equal to $\mathbb{R}^K_+$; for example, we could have $X = \mathbb{Z}^K_+$, the set of integral consumption points. For $x \in X$, the kth entry of x specifies the consumer's consumption of the kth good. A modeler makes observations of a consumer; at observation t, the consumer chooses the bundle $x_t \in X$, when the prices of the K goods are given by the vector $p_t \in \mathbb{R}^K_{++}$. Let $\mathcal{O}$ be a set of observations, consisting of (p_t, x_t), for $t = 1, \ldots, T$. A utility function $u : X \to \mathbb{R}$ is said to *rationalize the set of observations* $\mathcal{O}$ if x_t solves

$$\max_{x \in X} u(x) \text{ subject to } p_t \cdot x \leq p_t \cdot x_t. \tag{1}$$

The set $\mathcal{O}$ is said to obey the *generalized axiom of revealed preference* (GARP) if whenever there are observations (p_k, x_k) $(k = 1, 2, ..., n)$ in $\mathcal{O}$ satisfying

$$p_1 \cdot x_2 \leq p_1 \cdot x_1;\; p_2 \cdot x_3 \leq p_2 \cdot x_2;\; \ldots;\; p_{n-1} \cdot x_n \leq p_{n-1} \cdot x_{n-1};\; p_n \cdot x_1 \leq p_n \cdot x_n \tag{2}$$

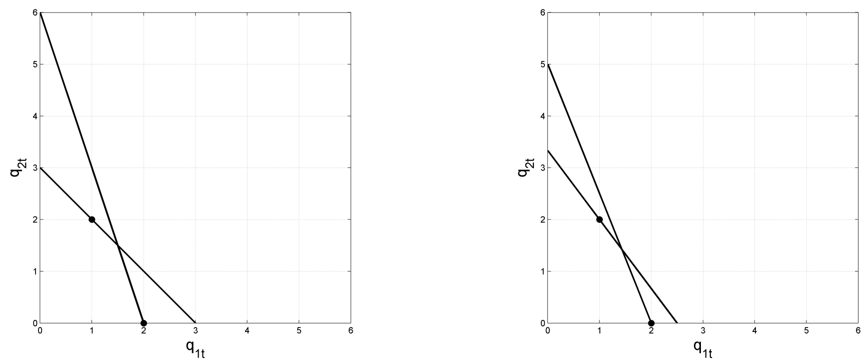
then all the inequalities have to be equalities. It is well-known and straightforward to check that if the set of observations are drawn from an agent maximizing a locally non-satiated utility function $U : X = \mathbb{R}^K_+ \to \mathbb{R}$, then the observations will obey GARP. Afriat's Theorem tells us the converse: if the observations obey GARP, then there is a strictly increasing[1] and concave utility function $U : X = \mathbb{R}^K_+ \to R$ that rationalizes that data.[2] The following two examples consider what happens when the consumption space is discrete rather than $\mathbb{R}^K_+$. In both examples, we assume that money is used for the purchase of two goods which can only be bought in whole units, so $X = \mathbb{Z}^2_+$.

Example 1. Suppose that in period 1, we observe $p_1 = (3, 3)$ and $x_1 = (1, 2)$, followed by $p_2 = (6, 2)$ and $x_2 = (2, 0)$ in period 2. This is depicted in Figure 1a. Plainly we have a violation of GARP since $p_1 \cdot x_1 > p_1 \cdot x_2$ and $p_2 \cdot x_2 > p_2 \cdot x_1$. And it is also the case that these observations are not compatible with the maximization of a strictly increasing utility function. Suppose to the contrary that the agent is maximizing such a utility function. Then Period 2's observation reveals that $(2, 0)$ is weakly preferred to $(1, 3)$ and (because the utility function is strictly increasing) $(1, 3)$ is strictly preferred to $(1, 2)$, so $(2, 0)$ is strictly preferred to $(1, 2)$. On the other hand, in period 1, $(1, 2)$ is chosen even though $(2, 0)$ is available, so we obtain a contradiction. Indeed, it is straightforward to see that we could make a stronger claim: with these two budget sets, *every* violation of GARP is incompatible with the maximization of a strictly increasing utility function.

Example 2. In Figure 1b, we first observe $p_1 = (4, 3)$ and $x_1 = (1, 2)$ in period 1, followed by $p_2 = (5, 2)$ and $x_2 = (2, 0)$ in period 2. Once again it is clear that GARP is violated. However, it is plain that these choices *are* compatible with rationality in the sense that there is a strictly increasing

[1] Formally, if $x' > x$ then $U(x') > U(x)$.

[2] For a short proof of this result see Fostel et al. (2004). A generalization of Afriat's Theorem to nonlinear budget sets can be found in Forges and Minelli (2009).



(a) Violation

(b) Non-violation

Figure 1: Revealed preference in a discrete consumption space

utility function defined on $\mathbb{Z}^2_+$ that could explain the data as solutions to (1). The crucial difference here is that (unlike the case above) we are not in position to say that x_2 is strictly preferred to x_1 because there is no affordable bundle (in the consumption space) in period 2, y, such that $y > x_1$. So these observations could be explained by some strictly increasing utility function that gives the same utility to x_1 and x_2.

In an experimental setting, it may be possible for the experimenter to ensure that budget sets are like those in Example 1. In that case, a violation of GARP can be legitimately considered a violation of rationality (for example, see Harbaugh, Krause, and Berry (2001)). However, in observational settings, budget sets are not prescribed by the observer and situations like that depicted in Example 2 could arise; here GARP is violated and yet the data are consistent with the maximization of a strictly increasing utility function.

Does this mean that we should drop or modify GARP when studying consumer choice over a discrete consumption space? The fundamental point we make in this Note is that that is *not* the case: while the observations depicted in Example 2 are consistent with a consumer solving (1), they are incompatible with a broader notion of rationality. This is because the consumer is spending more money than necessary to achieve the same level of utility: any utility function consistent with the observations in Example 2 must give the same utility to x_1 and x_2, and yet at each period, the consumer chooses to buy the bundle that is more costly. In the next section, we formalize this intuition and give the conditions under which price and demand observations from a discrete consumption space will still obey GARP.

3. GARP in observational data

We consider a consumer with the consumption space $X \times Y$, where $X \subset \mathbb{R}_+^K$ and $Y = \mathbb{R}_+$, and the utility function $u : X \times Y \to \mathbb{R}$. We assume that u has a weakly separable structure, i.e., there are functions $v : X \to \mathbb{R}$ and $\tilde{u} : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ such that $u(x, y) = \tilde{u}(v(x), y)$, and where $\tilde{u}$ is strictly increasing in both arguments. This last assumption means overall utility increases strictly with the sub-utility derived from consumption in X, $v(x)$, and also consumption y of the $(K+1)$th good. We shall refer to this last good as the *continuous good* since it could be consumed in infinitesimal quantities. The agent's problem is to

$$\max_{(x,y)\in X\times Y} u(x, y) = \tilde{u}(v(x), y) \text{ subject to } p \cdot x + qy \leq m, \tag{3}$$

where $q > 0$ is the price of the continuous good and $p \gg 0$ the price vector of bundles in X.[3]

[3] We assume, for simplicity, that there is just one continuous good. It is not hard to see that Proposition 1 goes through even when there are multiple goods apart from the K goods in X. What is crucial is that one of those goods can be consumed in continuous quantities and that the overall utility function $\tilde{u}$ is strictly increasing in that good.

The main result of this Note is Proposition 1 below. It says that so long as the agent is maximizing some overall utility function that includes a continuous good, then observations of prices and demand for the K goods in X will obey GARP, even when X is a discrete consumption space.

Proposition 1 *Suppose that the set of observations $\mathcal{O} = \{(p_t, x_t)\}_{1 \leq t \leq T}$, (of the price and demand for the K goods in X) are drawn from a consumer solving (3), with $m_t = p_t \cdot x_t$.*[4] *Then $\mathcal{O}$ obeys GARP.*

Proof: First we show that at any observation (p_t, x_t), the following properties hold: (i) if $p_t \cdot x_t = p_t \cdot x$ then $v(x_t) \geq v(x)$ and (ii) if $p_t \cdot x_t > p_t \cdot x$ then $v(x_t) > v(x)$. Property (i) says that x_t is utility-maximizing in the sense that it must have weakly higher (sub)utility than any bundle that costs the same while (ii) says that it is cost efficient, in the sense that if it costs more than some other bundle, then it must give higher (sub)utility. Assuming (i) and (ii), if (2) holds, then

$$v(x_1) \leq v(x_n) \leq v(x_{n-1}) \leq ... \leq v(x_2) \leq v(x_1),$$

so they must all be equal. GARP requires that we cannot have $p_t \cdot x_{t'} < p_t \cdot x_t$ in (2); this is true because it would imply (by (ii)) that $v(x_{t'}) < v(x_t)$.[5]

To prove (i), suppose $p_t \cdot x_t = p_t \cdot x$ but $v(x_t) < v(x)$. Then the bundle (x, y_t) is strictly preferred by the agent to (x_t, y_t) (where y_t is the (unobserved) choice of the continuous good made by the agent) since $\tilde{u}$ is strictly

[4] The observations do not include the price and the demand for the continuous good.

[5] Just as (i) alone is not sufficient to guarantee GARP (Example 2), so (ii) alone is also not sufficient. For example, suppose $X = \mathbb{Z}^2_+$, with $U(0,2) = U(1,0) = 3$ and $U(0,1) = 2$; clearly this utility function is strictly increasing. Suppose that at the price $(1, 1/2)$, we observe the consumer choosing $(1,0)$ and at price $(1,1)$, the consumer chooses $(0,1)$. These two observations are compatible with a consumer minimizing cost, subject to utility targets of 3 and 2 respectively, but they violate GARP.

increasing in the first argument; furthermore, the bundle (x, y_t) is also affordable at observation t, so we obtain a contradiction.

To prove (ii), suppose $p_t \cdot x_t > p_t \cdot x$ but $v(x_t) \leq v(x)$. Then the bundle $(x, y_t + [p_t \cdot (x_t - x)]/q_t)$ (where q_t is the price of the continuous good at period t) is strictly preferred by the agent to (x_t, y_t) since $\tilde{u}$ is strictly increasing in the second argument and it is also affordable at period t. In other words, because x_t costs more than the bundle x without giving greater utility, the agent is better off buying x and using the money saved to buy more of the continuous good. So once again we obtain a contradiction. **QED**

Note that Proposition 1 does *not* require v to be an increasing or concave function; the crucial assumption is that $\tilde{u}$ is strictly increasing in both arguments. Of course, given that $\mathcal{O}$ obeys GARP, then a straightforward application of Afriat's Theorem will guarantee the existence of a strictly increasing and concave utility function that rationalizes $\mathcal{O}$. This is stated formally in the next result, which is the converse of Proposition 1.

Proposition 2 *Suppose the set of observations $\mathcal{O} = \{(p_t, x_t)\}_{1 \leq t \leq T}$, (of the price and demand for the K goods in X) obeys GARP. Then there exists a function $V : X \to \mathbb{R}$ with the following properties:*
(a) *V rationalizes the data, i.e.,*

$$x_t \in \arg\max_{x \in X} V(x) \text{ subject to } p_t \cdot x \leq p_t \cdot x_t;$$

(b) *for all $x \in X$ with $p_t \cdot x < p_t \cdot x_t$, we have $V(x) < V(x_t)$;*
(c) *V admits an extension to $\mathbb{R}^K_+$ that is strictly increasing and concave (and hence is strictly increasing and concave in X);*[6]
(d) *given any $w > p_t \cdot x_t$ for all t, there is a real number $q_t > 0$ for every t*

[6] Concavity here means that that $V(\sum_{i=1}^m \alpha_i x_i) \geq \sum_{i=1}^m \alpha_i V(x_i)$ whenever x_i (for $i = 1, 2, ..., m$) and $\sum_{i=1}^m \alpha_i x_i$ are in X, where $\alpha_i > 0$ (for $i = 1, 2, ..., m$) and $\sum_{i=1}^m \alpha_i = 1$.

such that[7]

$$\left[x_t\,,\frac{(w-p_t\cdot x_t)}{q_t}\right]\in \arg\max_{(x,y)\in X\times\mathbb{R}_+} V(x)+y \text{ subject to } p_t\cdot x+q_t y\le w. \tag{4}$$

This proposition says (through properties (a) and (c)) that when $\mathcal{O}$ obeys GARP, there is a strictly increasing and concave utility function defined on the consumption space X that rationalizes the data; furthermore, with this utility function, any bundle in X that is strictly cheaper than the observed bundle will have strictly lower utility (property (b)). Property (d) says that the observations in $\mathcal{O}$ are consistent with a consumer maximizing an overall utility function defined on $X\times\mathbb{R}_+$; in other words, besides the K goods in X, the consumer also demands a continuous good (though the price and demand for this good are not observed). The utility function can be chosen to be additively separable over these two good categories, i.e., $U(x,y)=V(x)+y$. At each observation t, there is a price $q_t>0$ for the continuous good at which the bundle $(x_t,[w-p_t\cdot x_t]/q_t)$ maximizes utility within the budget.[8]

Proof of Proposition 2: Since GARP holds, Afriat's Theorem tells us that there is a strictly increasing and concave function $\bar{V}:\mathbb{R}^K_+\to\mathbb{R}$ such that x_t maximizes $\bar{V}(x)$ in the set $\{x\in\mathbb{R}^K_+ : p_t\cdot x\le p_t\cdot x_t\}$. Since $\bar{V}$ is strictly increasing, we must have $\bar{V}(x)<\bar{V}(x_t)$ for any x with $p_t\cdot x<p_t\cdot x_t$. Defining V as the restriction of $\bar{V}$ to X, it is clear that x_t maximizes $V(x)$ in the set $\{x\in X: p_t\cdot x\le p_t\cdot x_t\}$. So we have shown (a) to (c).

In fact, it is known that V can be chosen to have the following form:

$$V(x)=\min_{1\le t\le T}\{\phi_t+\lambda_t p_t\cdot(x-x_t)\} \tag{5}$$

[7] Note that the price of the continuous good q_t is allowed to vary with t. If it does not, then there exists V such that (4) holds for all t if and only if $\mathcal{O}$ obeys a property stronger than GARP called cyclical monotonicity (see Brown and Calsamiglia (2007)).

[8] Notice that there is some redundancy in the properties since we could in fact derive (a) and (b) from (c) and (d), but we think this is a clearer way of stating the result.

where $\lambda_t > 0$ and the scalars ϕ_t and λ_t are chosen in such a way that $V(x_t) = \phi_t$ (see Fostel et al. (2004)). If we set $q_t = 1/\lambda_t$, we obtain

$$\begin{aligned} V(x) + \frac{(w - p_t \cdot x)}{q_t} &\leq \phi_t + \lambda_t p_t \cdot (x - x_t) + \frac{(w - p_t \cdot x)}{q_t} \\ &\leq \phi_t + \frac{(w - p_t \cdot x_t)}{q_t} \\ &= V(x_t) + \frac{(w - p_t \cdot x_t)}{q_t}. \end{aligned}$$

In other words, (d) holds. **QED**

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